INTEGRABILITY FOR LINEAR GROWTH

SINA TURELI AND KHADIM WAR

Abstract. We prove that dominated splittings with linear growth are uniquely integrable.

1. Introduction and statement of result

Throughout this paper M denotes a three-dimensional compact manifold without boundary and φ is a C^2 diffeomorphism acting on it. The diffeomorphism φ is said to have *dominated splitting* if there are real numbers $0 < \lambda < 1 < \sigma$ and $C > 0$; and for every $x \in M$, a $D\varphi$-invariant continuous splitting of the tangent space of M at x

$$T_xM = E_x \oplus F_x \tag{1.1}$$

such that

$$\frac{\|D\varphi_x^k(v)\|}{\|D\varphi_x^k(w)\|} \leq C\lambda^k, \forall v \in E_x, w \in F_x, \text{ with } \|v\| = \|w\| = 1$$

and

$$\|D\varphi_x^k|_{F_x}\| \geq \sigma^k, k > 1.$$

The subspaces or subbundles E and F form a Hölder continuous distributions over M [2] which, in general, are not Lipschitz even if φ is C^2 or better [11].

An immersion $N \subset M$ is an *integral manifold* of a subbundle E if $T_xN = E_x$ at each point $x \in N$. A subbundle E is *integrable*, respectively *uniquely integrable*, if there exists an integral manifold, respectively a unique integral manifold, through every point of M. The integrability and unique integrability of a given invariant subbundle are classical questions for the study of dynamical systems. In general given a subbundle there two main obstacles for its integrability first to satisfy satisfy the Frobenius involutivity condition and second the lack of differentiability. This later obstacle is somehow common to most invariant subbundles under some dynamical system.

Nevertheless there are results for integrability under some dynamical conditions for invariant bundles. In the early 60's, people studied the so called Anosov diffeomorphisms which are characterized by the fact that the bundles E and F, which were referred as the stable E^s and unstable E^u bundles respectively, satisfy

$$\|D\varphi_x|_{E_x^s}\| < 1 < \|D\varphi_x|_{E_x^u}\|, \quad \text{for all} \quad x \in M. \tag{1.2}$$

For the Anosov diffeomorphisms, it is shown that the stable and unstable bundles are uniquely integrable [13] hence give arise to the stable and unstable foliations.

In the early 70's, M. Hirsch, C. Pugh and M. Shub in [13, 14]; M. Brin and Y. Pesin in [5, 6], in separate studies introduced the notion of *partially hyperbolic systems* in which they relax the contraction of the bundle E^s in the domination (1.2) by assuming that it splits into two subbundles

$$TM = E^s \oplus E^c \oplus E^u$$

where E^s and E^u satisfy (1.2) and E^c has an intermediate behaviour.

For partially hyperbolic systems, the unique integrability of the bundles $E^{sc} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$, which is referred as *dynamical coherence* is not guarantee mainly because of the central bundle. [22] gives an example of partially hyperbolic system for which the bundles have enough regularity but fail to satisfy the Frobenius involutivity condition. Also in [20] the authors present an example of partially hyperbolic diffeomorphism where where E^{cu} is integrable but not uniquely and somehow it satisfies the Frobenius "involutivity" condition but it is just Hölder continuous. Hence dynamical coherence for partially hyperbolic systems cannot be shown without any additional assumptions. In [4], the authors show that every absolute[1] partially hyperbolic system on the three torus has unique integrable E^{sc}, E^{cu} bundles. There are many other results for dynamical coherence that depend on the topology and geometry of the underlying manifold [3, 10, 18, 21].

In this paper we consider dominated splitting (1.1) where vectors in E can grow at most linearly, i.e there exists $C > 0$ such that

$$\|D\varphi^k v\| \leq Ck\|v\|, \quad \text{for all} \quad v \in E, k > 1. \tag{1.3}$$

This is our main Theorem.

Theorem 1.1. *Let $\varphi : M \to M$ be a C^2 diffeomorphism that has dominated splitting* (1.1). *If E has linear growth* (1.3) *then it is uniquely integrable.*

This linear growth assumption can be seen as a relaxation of the uniform contraction for Anosov diffeomorphism. On the other hand, for cocycles over Anosov diffeomorphisms, in [15], it is shown that vectors in E have polynomial growth in general. Moreover if $dim(M) = 3$, then E has linear growth with respect to the cocycles, however our result does not apply for cocycles in general.

As we pointed out above, the Frobenius Theorem was not so far useful for the problem of integrability in dynamical systems though there versions for lower regularity [12]. The novelty in our proof is based on a recent version of Frobenius Theorem for continuous bundles [17] which we also believe that it can be useful for certain problems of dynamical systems e.g the study of robust transitive diffeomorphisms [9].

2. Strategy

The proof of Theorem 1.1 is an application of the continuous version of Frobenius Theorem [17] which uses the following notion of involutivity.

Definition 2.1. A continuous distribution $\Delta = ker(\eta)$ on a three dimensional manifold M is uniformly asymptotically involutive on average if for every $x_0 \in M$, there exist a coordinate system $(x^1, x^2, x^3, \mathcal{U})$ around x_0, a sequence of C^1 differential 1-forms (η_k), a C^1 frame of $ker(\eta_k)$ of the form $X_k = \partial/\partial x^1 + a_k\partial/\partial x^3, Y_k = \partial/\partial x^2 + b_k\partial/\partial x^3$ (for some C^1 functions a_k, b_k) and $T > 0$ such that for every $k > 1$, $x \in \mathcal{U}$ and $|t| \leq T$ we have

- $\|\eta_k - \eta\|_x \to 0$
- $\|\eta_k \wedge d\eta_k\|_x \max\{\exp(\int_0^t d\eta_{k,1} \circ e^{\tau X^{(k)}}(x)), \exp(\int_0^t d\eta_{k,2} \circ e^{\tau Y^{(k)}}(x))\} \to 0$
- $\|\eta_k - \eta\|_x \max\{\exp(\int_0^t d\eta_{k,1} \circ e^{\tau X^{(k)}}(x)), \exp(\int_0^t d\eta_{k,2} \circ e^{\tau Y^{(k)}}(x))\} \to 0$

as $k \to \infty$, where $\eta_{k,i}, i = 1, 2$ are such that

$$d\eta_k = d\eta_{k,1}dx^1 \wedge dx^3 + d\eta_{k,2}dx^2 \wedge dx^3 + d\eta_{k,3}dx^1 \wedge dx^2$$

[1]the stabe bundle is uniformly dominated by the central which is uniformly dominated by the unstable

$$\|D\varphi_x v^s\| < \|D\varphi_y v^c\| < \|D\varphi_z v^u\|$$

for every $x, y, z \in M, v^\mu \in E^\mu, \|v^\mu\| = 1, \mu = s, c, u$.

and $e^{\tau X_k}(x), e^{\tau Y_k}(x)$ denotes the flow of X_k, Y_k through x at time τ.

Here is the continuous version of Frobenius Theorem

Theorem 2.2. [17] *A uniformly asymptotically involutive on average two dimensional distribution on a three dimensional manifold is uniquely integrable.*

For the rest of this section we are going to define the objects in Definition 2.1 and state the result which gives the asymptotic involutivity on average with respect to these objects.

Let $E^{(0)}$ be a C^1 two dimensional distribution transverse to F, for $k > 1$ and $x \in M$, we define

$$E^{(k)}_x = D\varphi^{-k}_{\varphi^k x} E^{(0)}_{\varphi^k x}. \tag{2.1}$$

The standard theory of dominated splitting in dynamical systems shows that the sequence of distributions $\{E^{(k)}\}_{k>1}$ is converging to E (we will prove it formally in Section 3).

We now fix some arbitrary point $x_0 \in M$ and a local chart $(\mathcal{U}_0, x^1, x^2, x^3)$ centered at x_0 and suppose that the invariant distribution E is transverse to the coordinate axes. Thus, by the convergence of $E^{(k)}$ to E, for k large enough the distribution $E^{(k)}$ is also transverse to the coordinate axes and then it can be spanned by vector fields X_k and Y_k of the form

$$X_k = \frac{\partial}{\partial x^1} + a_k \frac{\partial}{\partial x^3} \quad \text{and} \quad Y_k = \frac{\partial}{\partial x^2} + b_k \frac{\partial}{\partial x^3}. \tag{2.2}$$

Hence in this coordinate system we can write

$$\begin{aligned} E^{(k)} &= ker(\eta_k := dx^3 - a_k dx^1 - b_k dx^2) \\ E &= ker(\eta := dx^3 - a dx^1 - b dx^2) \end{aligned} \tag{2.3}$$

where a_k and b_k are C^1 functions converging to the continuous functions a and b respectively. The next result is the key estimation in this paper which gives the uniform asymptotic involutivity on average with respect to the choice of coordinate system above.

Proposition 2.3. *There exists $T > 0$ and $\mathcal{U} \subset \mathcal{U}_0$ such that for every $x \in \mathcal{U}$ and $|t| \le T$ we have*

$$\lim_{k\to 0} \|\eta_k \wedge d\eta_k\|_x \max\left[\exp(\int_0^t d\eta_{k,1} \circ e^{\tau X^{(k)}}(x)), \exp(\int_0^t d\eta_{k,2} \circ e^{\tau Y^{(k)}}(x))\right] = 0$$

and

$$\lim_{k\to 0} \|\eta_k - \eta\|_x \max\left[\exp(\int_0^t d\eta_{k,1} \circ e^{\tau X^{(k)}}(x)), \exp(\int_0^t d\eta_{k,2} \circ e^{\tau Y^{(k)}}(x))\right] = 0.$$

In all the estimations we will be using a generic constant $C > 0$ that does not depend on k.

3. Dominated splittings

In this section we will talk about general properties of dominated splittings. We start by the following proposition.

Proposition 3.1. *Let $\varphi : M \to M$ be a C^2 diffeomorphism admitting a continuous $D\varphi$-invariant dominated decomposition $E \oplus F$ as in (1.1). Let $E^{(0)}$ be a C^0 distribution transverse to F and let $\{E^{(k)}\}_{k\ge 1}$ be a sequence of C^0 distributions defined by (2.1). Then for every $x \in M$ we have*

$$E^{(k)}_x \to E_x \tag{3.1}$$

as $k \to \infty$ and there exists $C > 0$ such that for k large enough and $x \in M$ we have

$$\|D\varphi_x^k|_{E_x^{(k)}}\| \leq C\|D\varphi_x^k|_{E_x}\|.$$

Both statements of Proposition 3.1 follow by fairly elementary arguments from the domination condition (1.1). The first statement about the pointwise convergence in (3.1) is quite standard, we just sketch the argument here after recalling some basic notions and properties of partially hyperbolic systems (see [19] for details). First of all we introduce a metric, sometimes called an adapted metric, which orthogonalizes the splitting $E \oplus F$ that will simplify the calculations. For $v = v_E + v_F, w = w_E + w_F \in E \oplus F$ we define the Lyapunov metric by

$$< v, w >' = < v_E, w_E > + < v_F, w_F > .$$

Let $|\cdot|$ be the norm associated to the Lyapunov inner metric which satisfies $|v|_x^2 = |v_E|_x^2 + |v_F|_x^2$. One can show that $|\cdot|$ is equivalent to the Euclidean norm $\|\cdot\|$, i.e.

$$\theta\|v\| \leq |v| \leq \|v\| \tag{3.2}$$

for every $v \in TM$ where θ depends on the minimum angle between E and F (which is uniformly bounded below). Given a point $x \in M$, and $\alpha > 0$, we define the cone at x around E_x of angle α by

$$C^E(x, \alpha) = \{v = v_E + v_F \in E_x \oplus F_x : |v_F| < \alpha|v_E|\}$$

and similarly the cone around F_x

$$C^F(x, \alpha) = \{v = v_E + v_F \in E_x \oplus F_x : |v_E| < \alpha|v_F|\}.$$

For $k > 1$, it is easy to see that, due to invariance of the splitting and the domination condition (1.1), the cones are invariant

$$D\varphi_x^{-k}C^E(x, \alpha) \subset C^E(\varphi^{-k}x, \lambda^k\alpha) \quad \text{and} \quad D\varphi_x^k C^F(x, \alpha) \subset C^F(\varphi^k x, \lambda^k\alpha)$$

which immediately implies (3.1).

For the last estimation of Proposition 3.1, we also use the same cone estimation and the Lyapunov metric.

Proof of Proposition 3.1 last statement. Let $x \in M$ and $v \in E_x^{(k)}$ such that

$$|D\varphi_x^k|_{E_x^{(k)}}| = |D\varphi_x^k v|$$

Then writing $v = v_E + v_F \in E \oplus F$ and by the observation that $D\varphi^k v \in E \in C^E(\varphi^k x, \alpha)$ we have

$$|D\varphi_x^k|_{E_x^{(k)}}|^2 = |D\varphi_x^k v_E|^2 + |D\varphi_x^k v_F|^2 \leq (1 + \alpha^2)|D\varphi_x^k v_E|^2 \leq (1 + \alpha^2)|D\varphi_E^k|^2|v_E|^2$$

and the result follows using the equivalence between the norms (3.2). □

In the next lines we are going to deduce from the domination (1.1) some estimations related to the differential forms that define $E^{(k)}$ which will be usefull for proof of Proposition 2.3.

Let η_0 be a C^1 differential forms such that

$$E^{(0)} = ker(\eta_0)$$

then obviously for $k > 1$ we have

$$E^{(k)} = ker((\varphi^k)^*\eta_0)$$

where $(\varphi^k)^*\eta_0$ denotes naturally the pullback of the form η_0 by the diffeomorphism φ^k.

Assuming that we are given a coordinate systems $(x^1, x^1, x^3, \mathcal{U}_0)$ as in Section 2, we have the following

Lemma 3.2. *There exists $C > 0$ such that for k large enough and $x \in \mathcal{U}_0$ we have*

$$\frac{1}{C} \leq \frac{\|D\varphi_x^k|_{F_x}\|}{|\eta_0(D\varphi_x^k \frac{\partial}{\partial x^3})|} \leq C$$

Proof. Let $k > 1, x \in \mathcal{U}_0$ then obviously we have

$$|\eta_0(D\varphi_x^k \frac{\partial}{\partial x^3})| \leq \|\eta_0\| \|D\varphi_x^k\|.$$

Since $\frac{\partial}{\partial x^3}$ is uniformly transverse to sequence of distributions $E^{(k)}$ then there exists $\beta > 0$ such that $\frac{\partial}{\partial x^3}|_x \in C^F(x, \beta)$ which implies that $D\varphi_x^k \frac{\partial}{\partial x^3} \in C^F(\varphi^k x, \lambda^k \beta)$. Let $v_F \in F_x$ such that

$$|D\varphi_x^k|_{F_x}| = |D\varphi_x^k v_F|$$

Then we can write $v_F = v_E + c\frac{\partial}{\partial x^3}$ where $|c| \leq 1$ hence we have

$$|D\varphi_x^k v_F| \leq |D\varphi_x^k v_E| + |D\varphi_x^k \frac{\partial}{\partial x^3}| \leq \lambda^k \beta |D\varphi_x^k v_F| + |D\varphi_x^k \frac{\partial}{\partial x^3}|$$

which implies that

$$|D\varphi_x^k \frac{\partial}{\partial x^3}| \geq \frac{1}{1 - \lambda^k \beta} |D\varphi_x^k v_F|.$$

Since $ker(\eta_0)$ is transverse to F then for k large enough the minimum norm of η_0 restricted to the cone $C^F(\varphi^k x, \lambda^k \beta)$ is bounded away from 0 which implies that

$$|\eta_0(D\varphi_x^k \frac{\partial}{\partial x^3})| \geq \frac{1}{C} |D\varphi_x^k \frac{\partial}{\partial x^3}| \geq \frac{1}{C(1 - \lambda^k \beta)} |D\varphi_x^k|_{F_x}|.$$

We finally use the equivalence of norms (3.2) to get the result. □

In all the estimations below we will use the following notation

$$\|D\varphi^k|_E\| = \sup_{x \in M} \|D\varphi_x^k|_{E_x}\|.$$

Lemma 3.3. *There exists $T_0 > 0$ such that for every $|t| \leq T_0$ and $x \in \mathcal{U}_0$ if E has linear growth then we have*

$$\lim_{k \to \infty} \frac{\|D\varphi^k|_E\|^2}{\|D\varphi_x^k|_{F_x}\|} \exp(Ct\|D\varphi^k|_E\|) = 0$$

and

$$\lim_{k \to \infty} \frac{\|D\varphi^k|_E\|}{\|D\varphi_x^k|_{F_x}\|} \exp(Ct\|D\varphi^k|_E\|) = 0.$$

Proof. Let $x \in \mathcal{U}_0, k > 1$, if E has linear growth then using Remark **??** we have

$$\frac{\|D\varphi^k|_E\|^2}{\|D\varphi_x^k|_{F_x}\|} \exp(Ct\|D\varphi^k|_E\|) \leq C\frac{k^2}{\sigma^k} \exp(Ctk) = Ck^2 \left(\frac{e^{Ct}}{\sigma}\right)^k$$

and

$$\frac{\|D\varphi^k|_E\|}{\|D\varphi_x^k|_{F_x}\|} \exp(Ct\|D\varphi^k|_E\|) \leq C\frac{k}{\sigma^k} \exp(Ctk) = Ck \left(\frac{e^{Ct}}{\sigma}\right)^k.$$

Then we can choose $T_0 = \frac{\log \sigma}{2C}$ to get the desired result. □

Since the vector fields X_k, Y_k defined in (2.2) have bounded norms then there exist $T_1 > 0$ and $\mathcal{U} \subset \mathcal{U}_0$ such that for every $|t| < T_1, k > 1$ and $x \in \mathcal{U}$ we have

$$e^{tY_k}(x), e^{tX_k}(x) \in \mathcal{U}_0.$$

We define

$$T := \min\{T_0, T_1\}. \tag{3.3}$$

Lemma 3.4. *There exists $C > 0$ such that for every $x \in \mathcal{U}_0, k > 1$ we have*

$$\|\eta_k - \eta\|_x \leq C \frac{\|D\varphi_x^k|_E\|}{\|D\varphi_x^k|_F\|}$$

where η_k and η are defined in (2.3).

Proof. Let $x \in \mathcal{U}_0, k > 1$ by the definition of η_k and η in (2.3) we have

$$\|\eta_k - \eta\| = \|(a - a_k)dx^1 + (b - b_k)dx^2\| \leq \|X_k - X\| + \|Y_k - Y\| \tag{3.4}$$

where $X = \partial/\partial x^1 + a\partial/\partial x^3$ and $Y = \partial/\partial x^2 + b\partial/\partial x^3$.

Observe that since $E_x^{(k)} \in C^E(\varphi^k x, \frac{\|D\varphi_x^k|_E\|}{\|D\varphi_x^k|_F\|}\alpha)$ for some $\alpha > 0$ for all k, then we have

$$\|X_k - X\|_x, \|Y_k - Y\|_x \leq \alpha \frac{\|D\varphi_x^k|_E\|}{\|D\varphi_x^k|_F\|}.$$

Plugging this last inequality into (3.4) gives the desired bound. □

4. Integrability

This section is devoted to prove Theorem 1.1 which , as we said before in Section 2, follows from the continuous version of Frobenius Theorem 2.2. Thus we will just prove Proposition 2.3 which gives the uniform asymptotic involutivity on average as required in Theorem 2.2.

First of all we recall that the distributions $E^{(k)}$ and E can be written as kernel of the following forms

$$\eta_k = dx^3 - a_k dx^1 - b_k dx^2 \quad \text{and} \quad \eta = dx^3 - a dx^1 - b dx^2$$

respectively. Observe that by direct calculation we have

$$d\eta_k = \frac{\partial a_k}{\partial x^3} dx^1 \wedge dx^3 + \frac{\partial b_k}{\partial x^3} dx^2 \wedge dx^3 + (\frac{\partial a_k}{\partial x^2} - \frac{\partial b_k}{\partial x^1}) dx^1 \wedge dx^2$$

On the other hand we can write

$$d\eta_k = d\eta_{k,1} dx^1 \wedge dx^3 + d\eta_{k,2} dx^2 \wedge dx^3 + d\eta_{k,3} dx^1 \wedge dx^2$$

By comparing the terms of the two formulae for $d\eta_k$ we have

$$\frac{\partial b_k}{\partial x^3} = \eta_{2,k}.$$

Integrating along integral curve of Y_k we have

$$\exp\left(\int_0^t \frac{\partial b_k}{\partial x^3} \circ e^{\tau Y_k}(x) d\tau\right) = \exp\left(\int_0^t d\eta_{2,k} \circ e^{\tau Y_k}(x) d\tau\right) \tag{4.1}$$

for every $|t| \leq T, k > 1$ and $x \in \mathcal{U}$. By the same arguments we also have

$$\exp\left(\int_0^t \frac{\partial a_k}{\partial x^3} \circ e^{\tau X_k}(x) ds\right) d\tau = \exp\left(\int_0^t d\eta_{1,k} \circ e^{\tau X_k}(x) d\tau\right) \tag{4.2}$$

for every $|t| \leq T, k > 1$ and $x \in \mathcal{U}$.

Then to prove Proposition 2.3, we will focus on the estimation of the left hand side of (4.1)-(4.2) which has the following geometric interpretation

Proposition 4.1. *For every $k \geq 1, x \in \mathcal{U}$ and $|t| \leq T$, we have*

$$\| \left(e^{-tX_k}\right)_* \frac{\partial}{\partial x^3}|_x \| = \exp - \int_0^t \frac{\partial a_k}{\partial x^3} \circ e^{\tau X_k}(x) d\tau$$

and

$$\| \left(e^{-tY_k}\right)_* \frac{\partial}{\partial x^3}|_x \| = \exp - \int_0^t \frac{\partial b_k}{\partial x^3} \circ e^{\tau Y_k}(x) d\tau$$

where $\left(e^{-tX_k}\right)_*$ and $\left(e^{-tY_k}\right)_*$ denote the pushforward of the corresponding flows.

Proof. We will prove the statement for X_k, the proof for Y_k is symmetric. Let $k > 1, x \in \mathcal{U}$ and $|t| < T$, then by the definition of the pushforward [1], we have

$$\begin{aligned}\frac{d}{dt}((e^{-tX_k})_* \frac{\partial}{\partial x^3}|_x) &= \left(e^{-tX_k}\right)_* [X_k, \frac{\partial}{\partial x^3}]|_x \\ &= -\left(e^{-tX_k}\right)_* \frac{\partial a_k}{\partial x^3} \frac{\partial}{\partial x^3}|_x \\ &= -\frac{\partial a_k}{\partial x^3} \circ e^{tX_k}(x) \left(e^{tX_k}\right)_* \frac{\partial}{\partial x^3}|_x.\end{aligned}$$

Integrating both sides we get

$$\left(e^{-tX_k}\right)_* \frac{\partial}{\partial x^3}|_x = \left(\exp - \int_0^t \frac{\partial a_k}{\partial x^3} \circ e^{\tau X_k}(x) d\tau\right) \frac{\partial}{\partial x^3}|_x.$$

□

Corollary 4.2. *For every $x \in \mathcal{U}$ and $|t| \leq T$ we have*

$$\exp \int_0^t \frac{\partial a_k}{\partial x^3} \circ e^{\tau X_k}(x) d\tau = \| \left(e^{tX_k}\right)_* \frac{\partial}{\partial x^3}|_{e^{tX_k}(x)} \|$$

and

$$\exp \int_0^t \frac{\partial b_k}{\partial x^3} \circ e^{\tau Y_k}(x) d\tau = \| \left(e^{tY_k}\right)_* \frac{\partial}{\partial x^3}|_{e^{tY_k}(x)} \|.$$

Proof. We will just prove the first equality, the second one being analogous. Let $x \in \mathcal{U}$ and $|t| \leq T$, by Proposition 4.1 we have

$$\| \left(e^{tX_k}\right)_* \frac{\partial}{\partial x^3}|_{e^{tX_k}(x)} \| = \exp - \int_0^{-t} \frac{\partial b_k}{\partial x^3} \circ e^{(t+\tau)X_k}(x) d\tau$$

making the change of variables $t + \tau \to \tau$ we have

$$\exp - \int_0^{-t} \frac{\partial b_k}{\partial x^3} \circ e^{(t+\tau)X_k}(x) d\tau = \exp - \int_t^0 \frac{\partial b_k}{\partial x^3} \circ e^{\tau X_k}(x) d\tau = \exp \int_0^t \frac{\partial b_k}{\partial x^3} \circ e^{\tau X_k}(x) d\tau.$$

Substituting this last equality into the first one gives the desired result. □

Lemma 4.3. *For every $x \in \mathcal{U}$ and $|t| < T$ we have*

$$\|e_*^{tY_k} \frac{\partial}{\partial x^3}|_x\| \leq C \frac{\|D\varphi^k_{e^{-tY_k}(x)}\|}{\|D\varphi^k_x\|} \exp(C|t|\|D\varphi|_E\|)$$

$$\|e_*^{tX_k} \frac{\partial}{\partial x^3}|_x\| \leq C \frac{\|D\varphi^k_{e^{-tX_k}(x)}\|}{\|D\varphi^k_x\|} \exp(C|t|\|D\varphi|_E\|)$$

Proof. We prove the first inequality, the second one being analogous. Let $t \in [-T, T]$ and $x \in \mathcal{U}$, without loss of generality, we can assume that $t > 0$.

Let η_0 be a C^1 differential 1-form such that $E^{(0)} = ker(\eta_0)$, then obviously we have $E^{(k)} = ker((\varphi^k)^* \eta_0)$. For $\delta > 0$, consider the following curves:

- $\mathcal{Y}_1(s) = e^{(s-t)Y_k}(x), 0 \leq s \leq t$,
- $\mathcal{X}_1(s) = e^{s\frac{\partial}{\partial x^3}}(e^{-tY_k}(x)), 0 \leq s \leq \delta$,
- $\mathcal{Y}_2(s) = e^{sY_k}(\mathcal{X}_1(\delta)), 0 \leq s \leq t$,
- $\mathcal{X}_2(s) = e^{tY_k}(\mathcal{X}_1(s)), 0 \leq s \leq \delta$.

Observe $\Gamma = \mathcal{Y}_1 \cup \mathcal{X}_1 \cup \mathcal{Y}_2 \cup \mathcal{X}_2$ is a closed curve that bound a disk D which can be parametrized as follows

$$D(s_1, s_2) = e^{s_1 Y_k} \circ e^{s_2 \frac{\partial}{\partial x^3}}(x), \quad \text{for} \quad 0 \leq s_1 \leq t, 0 \leq s_2 \leq \delta$$

First we remark that by the Stocke's Formula we have

$$\int_\Gamma (\varphi^k)^*\eta_0 = \int_{\varphi^k(D)} d\eta_0 \tag{4.3}$$

We are going to estimate the two terms separately. By linearity of the integral we have

$$\int_\Gamma (\varphi^k)^*\eta_0 = \int_{\mathcal{Y}_1} (\varphi^k)^*\eta_0 + \int_{\mathcal{X}_1} (\varphi^k)^*\eta_0 + \int_{\mathcal{Y}_2} (\varphi^k)^*\eta_0 + \int_{\mathcal{X}_2} (\varphi^k)^*\eta_0.$$

Since $\mathcal{Y}_1$, and $\mathcal{Y}_2$ are tangent to $E^{(k)} = ker((\varphi^k)^*\eta_0)$ then

$$\int_{\mathcal{Y}_1} (\varphi^k)^*\eta_0 = \int_{\mathcal{Y}_2} (\varphi^k)^*\eta_0 = 0.$$

Hence we get

$$\int_\Gamma (\varphi^k)^*\eta_0 = \int_0^\delta \eta_0(\varphi^k_* h_k(t,.)\frac{\partial}{\partial x^3})_{\varphi^k\circ\mathcal{X}_2(s)ds} - \int_0^\delta \eta_0(\varphi^k_* \frac{\partial}{\partial x^3})_{\varphi^k\circ\mathcal{X}_1(s)ds}. \tag{4.4}$$

Where h_k is defined by

$$e^{tY_k}_* \frac{\partial}{\partial x^3}|_x = h_k(t,x)\frac{\partial}{\partial x^3}|_x$$

which implies that

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_\Gamma (\varphi^k)^*\eta_0 = \eta_0(\varphi^k_* h_k(t,.)\frac{\partial}{\partial x^3})_{\varphi^k\circ\mathcal{X}_2(0)} - \eta_0(\varphi^k_* \frac{\partial}{\partial x^3})_{\varphi^k\circ\mathcal{X}_1(0)} \tag{4.5}$$

On the other hand using the parametrization of D we have

$$\begin{aligned}\int_{\varphi^k(D)} d\eta_0 &= \int_0^\delta\int_0^t d\eta_0(\frac{\partial\varphi^k\circ D}{\partial s_1}, \frac{\partial \varphi^k\circ D}{\partial s_2})_{\varphi^k\circ D(s_1,s_2)}ds_1ds_2\\ &= \int_0^\delta\int_0^t d\eta_0(\varphi^k_* Y_k, \varphi^k_* e^{s_1Y_k}_*\frac{\partial}{\partial x^3})_{\varphi^k\circ D(s_1,s_2)}ds_1ds_2\\ &= \int_0^\delta\int_0^t d\eta_0(\varphi^k_* Y_k, \varphi^k_* h_k(s_1,.)\frac{\partial}{\partial x^3})_{\varphi^k\circ D(s_1,s_2)}ds_1ds_2\end{aligned}$$

Which also implies that

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{\varphi^k(D)} d\eta_0 = \int_0^t d\eta_0(\varphi^k_* Y_k, \varphi^k_* h_k(s_1,.)\frac{\partial}{\partial x^3})_{\varphi^k\circ D(s_1,0)}ds_1 \tag{4.6}$$

And substituting Equations (4.5) and (4.6) into (4.3) we have

$$\eta_0(\varphi^k_* h_k(t,.)\frac{\partial}{\partial x^3})_{\varphi^k\circ\mathcal{X}_2(0)} - \eta_0(\varphi^k_* \frac{\partial}{\partial x^3})_{\varphi^k\circ\mathcal{X}_1(0)} = \int_0^t d\eta_0(\varphi^k_* Y^{(k)}, \varphi^k_* h_k(s_1,.)\frac{\partial}{\partial x^3})_{\varphi^k\circ D(s_1,0)}ds_1$$

By the observation that $\mathcal{Y}_1(.) = D(.,0)$ we can write

$$\begin{aligned}\eta_0(\varphi^k_* h_k(t,.)\frac{\partial}{\partial x^3})_{\varphi^k\circ\mathcal{X}_2(0)} - \eta_0(\varphi^k_* \frac{\partial}{\partial x^3})_{\varphi^k\circ\mathcal{X}_1(0)} &\leq \left(C\sup_{x\in M}\|D\varphi^k_x|_{E^{(k)}}\|\int_0^t \|\varphi^k_* h_k(s_1,.)\frac{\partial}{\partial x^3}|_{\varphi^k\circ\mathcal{Y}_1(s_1)}\|ds_1\right)\\ &\leq \left(C\sup_{x\in M}\|D\varphi^k_x|_{E^{(k)}}\|\int_0^t |\eta_0(\varphi^k_* h_k(s_1,.)\frac{\partial}{\partial x^3})|_{\varphi^k\circ\mathcal{Y}_1(s_1)}\right)\end{aligned}$$

where the last inequality is given by Lemma 3.2.

We finally use the Gronwall inequality to get

$$|\eta_0(\varphi^k_* h_k(t,.)\frac{\partial}{\partial x^3})_{\varphi^k\circ\mathcal{X}_2(0)}| \leq |\eta_0(\varphi^k_* \frac{\partial}{\partial x^3})_{\varphi^k\circ\mathcal{X}_1(0)}|\exp(Ct\sup_{x\in M}\|D\varphi^k_x|_{E^{(k)}}\|)$$

then we have

$$|h_k(t,x)| \leq \frac{|\eta_0(\varphi^k_*\frac{\partial}{\partial x^3})_{\varphi^k\circ\mathcal{X}_1(0)}|}{|\eta_0(\varphi^k_*\frac{\partial}{\partial x^3})_{\varphi^k\circ\mathcal{X}_2(0)}|}\exp(Ct\sup_{x\in M}\|D\varphi^k_x|_{E^{(k)}}\|)$$

The result follows using Lemma 3.2 and Proposition 3.1. □

Lemma 4.4. *There exists $C > 0$ such that for every $x \in \mathcal{U}_0, k > 1$ we have*

$$\|\eta_k \wedge d\eta_k\|_x \leq C\|[X_k, Y_k](x)\|$$

Proof. Let $x \in \mathcal{U}_0, k > 1$, observe that since X_k, Y_k and $\frac{\partial}{\partial x^3}$ form a basis of the tangent space T_xM then it is enough to show that

$$|\eta_k \wedge d\eta_k(X^{(k)}, Y^{(k)}, \frac{\partial}{\partial x^3})|_x \leq C\|[X_k, Y_k](x)\|.$$

By the definition of X_k, Y_k we have $\eta_k([X_k, Y_k]) = c_k\eta_k(\partial/\partial x^3)$ and therefore $c_k = \eta_k([X_k, Y_k])$ and in particular

$$\|[X_k, Y_k]\| = |\eta_k([X_k, Y_k])|.$$

Since $X_k, Y_k \in \ker(\eta_k)$, we have $\eta_k(X_k) = \eta_k(Y_k) = 0$ and the "Cartan formula" gives

$$d\eta_k(X_k, Y_k) = X_k(\eta_k(Y_k)) - Y_k(\eta_k(X_k)) - \eta_k([X_k, Y_k]) = -\eta_k([X_k, Y_k]). \tag{4.7}$$

On the other hand, we have

$$\eta_k \wedge d\eta_k\left(\frac{\partial}{\partial x^3}, X_k, Y_k\right) = \eta_k\left(\frac{\partial}{\partial x^3}\right)d\eta_k(X_k, Y_k) = d\eta_k(X_k, Y_k).$$

Substituting into the equations above we then get

$$\|[X_k, Y_k]\| = |\eta_k([X_k, Y_k])| = |d\eta_k([X_k, Y_k])| = \left|\eta_k \wedge d\eta_k\left(\frac{\partial}{\partial x^3}, X_k, Y_k\right)\right|$$

which concludes the proof according to the first observation. □

Proposition 4.5. *There exists $C > 0$ such that for every $k > 1$ and $x \in \mathcal{U}_0$ we have*

$$\|[X_k, Y_k](x)\| \leq C\frac{\|D\varphi^k_x|_{E_x}\|^2}{\|D\varphi^k_x|_{F_x}\|}.$$

Proof. Let $x \in \mathcal{U}$ and $k > 1$, again since $[X_k, Y_k] = c_k\partial/\partial x^3$ and therefore

$$\|[X_k, Y_k](x)\| = |c_k(x)| = \frac{|(\varphi^k)^*\eta_0([X_k, Y_k])|_x|}{|(\varphi^k)^*\eta_0(\frac{\partial}{\partial x^3})_x|} \leq C\frac{|(\varphi^k)^*\eta_0([X_k, Y_k])|_x|}{|\eta_0(D\varphi^k_x\frac{\partial}{\partial x^3})|}. \tag{4.8}$$

On the other hand using Cartan formula we have

$$|(\varphi^k)^*\eta_0([X_k, Y_k])|_x| = |d\eta_0(D\varphi^k_xX_k, D\varphi^k_xY_k)| \leq \|d\eta_0\|\|X_k\|\|Y_k\|\|D\varphi^k_x|_{E^{(k)}_x}\|^2 \tag{4.9}$$

Substituting (4.9) into (4.8) gives

$$\|[X_k, Y_k](x)\| \leq \|d\eta_0\|\|X_k\|\|Y_k\|\frac{\|D\varphi^k_x|_{E^{(k)}_x}\|^2}{||\eta_0(D\varphi^k_x\frac{\partial}{\partial x^3})|}.$$

Hence using the fact that X_k and Y_k have bounded norm, Proposition 3.1 and Lemma 3.2 we get the desired bound. □

We can finally prove Proposition 2.3.

Proof of Proposition 2.3. Let $\mathcal{U}$ and T given in Section 3 (cf.(3.3)).

Let $x \in \mathcal{U}, k > 1$ and $|t| < T$, by Corollary 4.2 and Lemma 4.3

$$\begin{aligned}\exp\int_0^t \frac{\partial a_k}{\partial x^3} \circ e^{\tau X_k}(x)d\tau &= \|\left(e^{tX_k}\right)_*\frac{\partial}{\partial x^3}|_{e^{tX_k}(x)}\| \\ &\leq C\frac{\|D\varphi^k_x|_F\|}{\|D\varphi^k_{e^{tX_k}(x)}|_F\|}\exp(tC\|D\varphi^k|_E\|)\end{aligned} \tag{4.10}$$

Also by Lemma 4.5 we have

$$\|[X_k, Y_k](x)\| \exp \int_0^t \frac{\partial a_k}{\partial x^3} \circ e^{\tau X_k}(x) d\tau \le C \frac{\|D\varphi_x^k|_{E_x}\|^2}{\|D\varphi^k_{e^{tX_k}(x)}|_F\|} \exp(tC\|D\varphi^k|_E\|)$$

Thus using (4.2) and Lemma 4.4 we have

$$\|\eta_k \wedge d\eta_k\|_x \exp\left(\int_0^t d\eta_{1,k} \circ e^{\tau X_k}(x) d\tau\right) \le C \frac{\|D\varphi^k|_E\|^2}{\|D\varphi^k_{e^{tX_k}(x)}\|} \exp(Ct\|D\varphi^k|_E\|)$$

Finaly by the assumption of linear growth for vectors in E and Lemma 3.3 we have that

$$\lim_{k\to\infty} \|\eta_k \wedge d\eta_k\|_x \exp\left(\int_0^t d\eta_{2,k} \circ e^{\tau X_k}(x) d\tau\right) = 0.$$

By exactly same arguments we have

$$\lim_{k\to\infty} \|\eta_k \wedge d\eta_k\|_x \exp\left(\int_0^t d\eta_{2,k} \circ e^{\tau Y_k}(x) d\tau\right) = 0.$$

Similarly, using Lemma 3.4 and (4.2), we have

$$\|\eta_k - \eta\|_x \exp\left(\int_0^t d\eta_{1,k} \circ e^{\tau X_k}(x) d\tau\right) \le C \frac{\|D\varphi_x^k|_E\|}{\|D\varphi^k_{e^{tY_k}(x)}\|} \exp(Ct\|D\varphi^k|_E\|)$$

Again by the assumption of linear growth of vectors in E and Lemma 3.3 we have

$$\lim_{k\to 0} \|\eta_k - \eta\|_x \exp\left(\int_0^t d\eta_{1,k} \circ e^{\tau X_k}(x) d\tau\right) = 0$$

again the same arguments give

$$\lim_{k\to\infty} \|\eta_k - \eta\|_x \exp\left(\int_0^t d\eta_{2,k} \circ e^{\tau Y_k}(x) d\tau\right) = 0$$

which concludes the proof. □

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Sina Tureli

Abdus Salam International Centre for Theoretical Physics (ICTP), Strada Costiera 11, Trieste, Italy and International School for Advanced Studies (SISSA), Via Bonomea 265, Trieste

Email address: sinatureli@gmail.com

Khadim War

Abdus Salam International Centre for Theoretical Physics (ICTP), Strada Costiera 11, Trieste, Italy and International School for Advanced Studies (SISSA), Via Bonomea 265, Trieste

Email address: kwar@ictp.it